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The branching rule for the restriction from SO(7) to G_2

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Abstract. Explicit expressions for the characters of irreducible representations of SO(7), G_2 and SU(3) are used to derive the branching rule appropriate to the restriction from SO(7) to G_2. The final result is expressed in three ways: (i) by means of a generating formula; (ii) in terms of a branching multiplicity diagram consisting in general of a set of nested heptagons; and (iii) through an explicit and simple formula for the multiplicity with which any irreducible representation of G_2 appears as a constituent of any irreducible representation of SO(7). A remarkable and unexpected connection with the weight multiplicity diagrams of SU(3) is pointed out.

1. Introduction

Since the discovery by Racah (1949) of the relevance of the group G_2 to the problem of labelling states of the atomic f-shell the properties of the representations of G_2 have received a great deal of attention. A very useful form of the branching rule for the restriction from SO(7) to G_2 has been given by Judd (1963), and a number of simplifications appropriate to quite wide classes of special cases have been pointed out by Wybourne (1972) along with a conjecture applicable to the most general case.

It is the purpose of this paper to use explicit forms for the characters of irreducible representations of SO(7), G_2 and SU(3) to derive what is in some sense the best possible form of the branching rule, and to show its connection with the weight diagrams of SU(3).

2. Characters of SO(7), G_2 and SU(3)

The irreducible representations of the groups SO(7), G_2 and SU(3) may be denoted by $[\lambda_1\,\lambda_2\,\lambda_3]$, $\{\mu_1\,\mu_2\}$ and $\{\nu_1\,\nu_2\}$ respectively, where λ_i for $i=1,2,3$ are either all integers or all half-odd integers satisfying $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant 0$, μ_i for $i=1,2$ are integers satisfying $\mu_1\geqslant\mu_2\geqslant 0$, and ν_i for $i=1,2$ are integers satisfying $\nu_1\geqslant\nu_2\geqslant 0$.

In terms of real class parameters ϕ_i, θ_i and ψ_i the characters of these representations may be written in the forms:

$$\text{SO(7):}\qquad \chi^{[\lambda_1\,\lambda_2\,\lambda_3]}=\frac{\sum_{\pi\in S_3}(-1)^{\pi}(x_{\pi_1}^{\frac{5}{2}+\lambda_1}-x_{\pi_1}^{-\frac{5}{2}-\lambda_1})(x_{\pi_2}^{\frac{3}{2}+\lambda_2}-x_{\pi_2}^{-\frac{3}{2}-\lambda_2})(x_{\pi_3}^{\frac{1}{2}+\lambda_3}-x_{\pi_3}^{-\frac{1}{2}-\lambda_3})}{\sum_{\pi\in S_3}(-1)^{\pi}(x_{\pi_1}^{\frac{5}{2}}-x_{\pi_1}^{-\frac{5}{2}})(x_{\pi_2}^{\frac{3}{2}}-x_{\pi_2}^{-\frac{3}{2}})(x_{\pi_3}^{\frac{1}{2}}-x_{\pi_3}^{-\frac{1}{2}})},\tag{2.1}$$

where

$$x_1=\exp \mathrm{i}\phi_1,\qquad x_2=\exp \mathrm{i}\phi_2,\qquad x_3=\exp \mathrm{i}\phi_3.\tag{2.2}$$

$$G_2: \qquad \chi^{\{\mu_1\ \mu_2\}} = \frac{\sum_{\pi\in S_3}(-1)^{\pi}(x_{\pi_1}^{2+\mu_1}x_{\pi_2}^{0}x_{\pi_3}^{-1-\mu_2}+x_{\pi_1}^{-2-\mu_1}x_{\pi_2}^{0}x_{\pi_3}^{1+\mu_2})}{\sum_{\pi\in S_3}(-1)^{\pi}(x_{\pi_1}^{2}x_{\pi_2}^{0}x_{\pi_3}^{-1}+x_{\pi_1}^{-2}x_{\pi_2}^{0}x_{\pi_3}^{1})}, \tag{2.3}$$

where

$$x_1 = \exp \mathrm{i}\frac{1}{2\sqrt{3}}\theta_1, \qquad x_2 = \exp\left(-\frac{1}{4\sqrt{3}}\theta_1+\frac{1}{4}\theta_2\right), \qquad x_3 = \exp \mathrm{i}\left(-\frac{1}{4\sqrt{3}}\theta_1-\frac{1}{4}\theta_2\right). \tag{2.4}$$

$$\mathrm{SU}(3): \qquad \chi^{\{\nu_1\ \nu_2\}} = \frac{\sum_{\pi\in S_3}(-1)^{\pi}x_{\pi_1}^{2+\nu_1}x_{\pi_2}^{1+\nu_2}x_{\pi_3}^{0}}{\sum_{\pi\in S_3}(-1)^{\pi}x_{\pi_1}^{2}x_{\pi_2}^{1}x_{\pi_3}^{0}} \tag{2.5}$$

where

$$x_1 = \exp \mathrm{i}\left(\frac{1}{2\sqrt{3}}\psi_1+\frac{1}{6}\psi_2\right), \qquad x_2 = \exp \mathrm{i}\left(-\frac{1}{2\sqrt{3}}\psi_1+\frac{1}{6}\psi_2\right), \qquad x_3 = \exp \mathrm{i}\left(-\frac{1}{3}\psi_2\right). \tag{2.6}$$

In these formulae π denotes the permutation of S_3 mapping (1 2 3) to $(\pi_1\ \pi_2\ \pi_3)$, and $(-1)^{\pi}$ its parity. The formulae (2.1) and (2.5) are well known and have been derived for example by Judd (1963). The formula (2.3) may be derived in the same way. The important point to note is that in carrying out the derivation of both (2.3) and (2.5) it is useful to parametrise the associated root spaces of G_2 and SU(3) using the parameters (2.2) of the root space of SO(7) subject to the constraint:

$$G_2 \text{ and } \mathrm{SU}(3): \qquad x_1\,x_2\,x_3 = 1. \tag{2.7}$$

This corresponds to the use of the triangular coordinate systems for the root spaces of both G_2 and SU(3). The particular parametrisations (2.4) and (2.6) correspond to the usual Cartesian coordinate systems for these root spaces introduced by Behrends *et al* (1962). However having pointed out this correspondence it should be stressed that the use of (2.2) in the interpretation of all three character formulae (2.1), (2.3) and (2.5) has a number of advantages. Firstly it makes manifest the Weyl symmetry group. In the case of SO(7) this group is the group of permutations and arbitrarily chosen sign changes of ϕ_1, ϕ_2 and ϕ_3; in the case of G_2 it is the group of permutations and simultaneous sign changes of ϕ_1, ϕ_2, and ϕ_3, whilst in the case of SU(3) it is simply the group of permutations of ϕ_1, ϕ_2, and ϕ_3. Secondly the embeddings of G_2 in SO(7) and of SU(3) in G_2 are defined precisely by the use of the same parameters x_1, x_2 and x_3 in all three character formulae. In order to derive the corresponding branching rules it is therefore only necessary to write the characters (2.1) and (2.3) as linear combinations of the characters (2.3) and (2.5) respectively, making use of the constraint (2.7). In deriving branching rules in this direct way it is possible that inadmissible characters may arise which may be eliminated by the use of the modification rules:

$$G_2: \qquad \chi^{\{\mu_1\ \mu_2\}} = -\chi^{\{\mu_2-1,\mu_1+1\}} = -\chi^{\{\mu_1+\mu_2+1,-\mu_2-2\}}, \tag{2.8}$$

$$\mathrm{SU}(3): \qquad \chi^{\{\nu_1,\nu_2\}} = -\chi^{\{\nu_2-1,\nu_1+1\}} = -\chi^{\{\nu_1-\nu_2-1,-\nu_2-2\}}. \tag{2.9}$$

These follow directly from (2.3) and (2.5) together with (2.7).

3. Weight diagrams of SU(3)

Digressing temporarily, the weight diagrams of irreducible representations of SU(3) may be constructed by expanding the characters (2.5) in the form:

$$\mathrm{SU(3)}: \qquad \chi^{\{\nu_1\,\nu_2\}} = \sum_{n_1,n_2,n_3} K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)} \exp \mathrm{i}(n_1\phi_1 + n_2\phi_2 + n_3\phi_3), \tag{3.1}$$

where the coefficient $K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)}$ is the multiplicity of the weight $(n_1\,n_2\,n_3)$ in the representation $\{\nu_1\,\nu_2\}$. The weight diagram is the point set in which each point specified by the triangular coordinates $(n_1\,n_2\,n_3)$ is assigned the multiplicity $K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)}$. Since (2.5) defines a Schur function (Littlewood 1940) it follows that this multiplicity is just the number of standard Young tableaux with row lengths $(\nu_1\,\nu_2)$ containing n_1 ones, n_2 twos and n_3 threes arranged to be non-decreasing across rows and strictly increasing down columns. The enumeration of such tableaux leads to the generating formula:

$$\mathrm{SU(3)}: \qquad \chi^{\{\nu_1\,\nu_2\}} = \sum_{\alpha_3=0}^{\nu_1-\nu_2} \sum_{\alpha_1=0}^{\nu_2} \sum_{\alpha_2=0}^{\alpha_1+\alpha_3} \exp \mathrm{i}[(\nu_2-\alpha_1+\alpha_2)\phi_1 + (\nu_2-\alpha_2+\alpha_3)\phi_2 + (\nu_1-\nu_2+\alpha_1-\alpha_3)\phi_3] \tag{3.2}$$

and hence to the explicit formula

$$K^{\{\nu_1\,\nu_2\}}_{(n_1\,n_2\,n_3)} = 1 + \min(\nu_1-\nu_2,\, \nu_2,\, \nu_1-n_1,\, \nu_1-n_2,\, \nu_1-n_3,\, n_1,\, n_2,\, n_3) \tag{3.3}$$

where

$$\min(\kappa_1, \kappa_2, \ldots) = \begin{cases} \text{minimum of } \kappa_1, \kappa_2, \ldots & \text{if } \kappa_i \geqslant 0 \text{ for all } i \\ -1 & \text{if } \kappa_i < 0 \text{ for any } i. \end{cases} \tag{3.4}$$

This generalises the result of Wigner (1937) in a manner which makes explicit the symmetry of the multiplicity with respect to permutations of $(n_1\,n_2\,n_3)$.

The corresponding weight diagram has the well known structure of a set of nested hexagons with the multiplicity increasing in steps of 1 from the value 1 on the boundary hexagon until the value $1+\min(\nu_1-\nu_2, \nu_2)$ is reached at which state the 'hexagon' is in fact either a triangle or a single point. The multiplicity at each point within or on the triangle takes on this same maximum value. The arguments of min(. . .) in (3.3) are just the lengths $\nu_1-\nu_2$ and ν_2 of alternate sides of the boundary hexagon and the distances of an arbitrary point specified by $(n_1\,n_2\,n_3)$ from the six edges of this hexagon.

This structure is exemplified in the case of the representation $\{\nu_1\,\nu_2\} = \{6\;4\}$ by the weight diagram of figure 1.

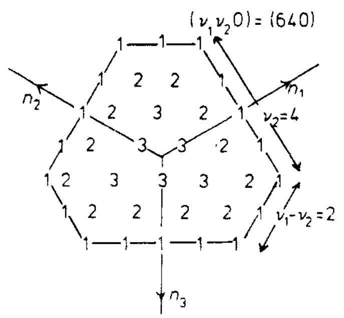

Figure 1. SU(3) weight diagram for the irreducible representation $\{\nu_1\,\nu_2\} = \{6\;4\}$. The numbers displayed are the multiplicities $K^{\{6\,4\}}_{(n_1\,n_2\,n_3)}$.

4. Branching rule for $\mathrm{SO}(7)\downarrow G_2$

Returning to the properties of G_2 the branching multiplicity diagram appropriate to the restriction from SO(7) to G_2 may be constructed, as pointed out already, by expanding the characters (2.1) in the form:

$$\mathrm{SO}(7)\downarrow G_2:\quad \chi^{[\lambda_1\,\lambda_2\,\lambda_3]}=\sum_{\mu_1\geqslant\mu_2\geqslant 0} B^{[\lambda_1\,\lambda_2\,\lambda_3]}_{\{\mu_1\,\mu_2\}}\,\chi^{\{\mu_1\,\mu_2\}}. \tag{4.1}$$

The branching multiplicity diagram is the point set in which each point specified by oblique coordinates $(\mu_1\,\mu_2)$ is assigned the branching multiplicity $B^{[\lambda_1\,\lambda_2\,\lambda_3]}_{\{\mu_1\,\mu_2\}}$.

Manipulating (2.1) and using the constraint (2.7) yields the result due to Judd (1963):

$$\chi^{[\lambda_1\,\lambda_2\,\lambda_3]}=\sum_{\rho_1=\lambda_2}^{\lambda_1}\sum_{\rho_2=\lambda_3}^{\lambda_2}\sum_{\rho_3=-\lambda_3}^{\lambda_3}\left(\chi^{\{\rho_1-\rho_3,\rho_2+\rho_3\}}+\chi^{\{\rho_2-\rho_3+1,\rho_1-\rho_2\}}\right) \tag{4.2}$$

where λ_i and ρ_i for $i=1,2,3$ are either all integers or all half-odd integers. This formula may conveniently be re-written as:

$$\begin{aligned}\chi^{[\lambda_1\,\lambda_2\,\lambda_3]}=&\sum_{\beta_1=0}^{\lambda_1-\lambda_2}\sum_{\beta_2=0}^{2\lambda_3}\sum_{\beta_3=0}^{\lambda_2-\lambda_3}\chi^{\{\lambda_2+\lambda_3+\beta_1-\beta_2,\lambda_2-\lambda_3+\beta_2-\beta_3\}}\\ &+\sum_{\gamma_1=0}^{2\lambda_3}\sum_{\gamma_2=0}^{\lambda_2-\lambda_3}\sum_{\gamma_3=0}^{\lambda_1-\lambda_2}\chi^{\{\lambda_1-\lambda_3-1+\gamma_1-\gamma_2,\lambda_1-\lambda_2+\gamma_2-\gamma_3\}}\end{aligned} \tag{4.3}$$

where each term gives rise, as a point set, to a set of nested hexagons in which the multiplicities increase in steps of 1 from the value 1 on the boundary hexagon until the value $1+\min(\lambda_1-\lambda_2,\,\lambda_2-\lambda_3,\,2\lambda_3)$ is reached, at which stage the 'hexagon' is either a parallelogram or a line or a single point. The multiplicity at each point within or on the parallelogram is this same maximum value. This structure is well illustrated by the example $[\lambda_1\,\lambda_2\,\lambda_3]=[11\,4\,2]$ as shown in figure 2.

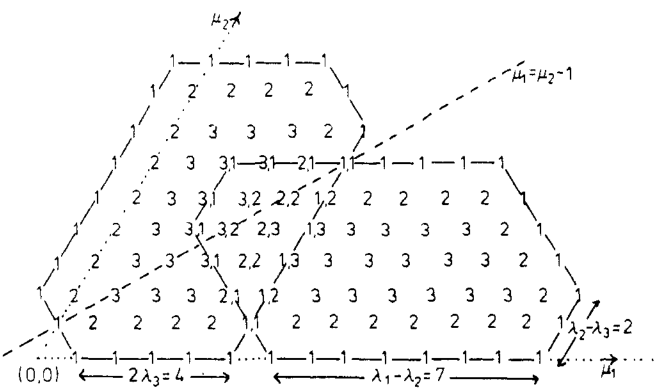

Figure 2. $\mathrm{SO}(7)\downarrow G_2$ branching multiplicity diagram for the irreducible representation $[\lambda_1\,\lambda_2\,\lambda_3]=[11\,4\,2]$ illustrating the two sets of nested hexagons, including inadmissible terms for which $\mu_1<\mu_2$.

Making use of (2.8) to eliminate inadmissible characters for which $\mu_1 < \mu_2$ corresponds to reflecting in the line $\mu_1 = \mu_2 - 1$ and subtracting the contribution to the multiplicity. This leads, for the above example, to the result exhibited in figure 3.

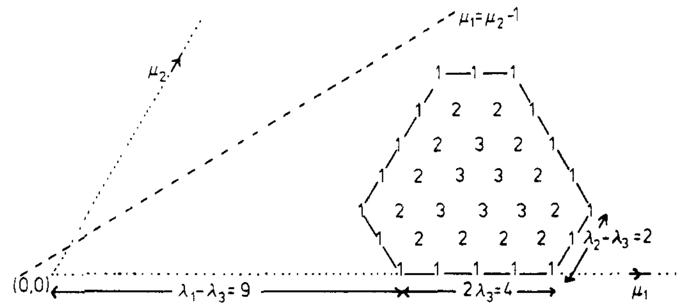

Figure 3. SO(7)$\downarrow G_2$ branching multiplicity diagram for the irreducible representation $[\lambda_1\,\lambda_2\,\lambda_3] = [11\;4\;2]$ after the application of the modification rules corresponding to reflection in the line $\mu_1 = \mu_2 - 1$. The numbers displayed are the branching multiplicities $B^{[11\,4\,2]}_{\{\mu_1\,\mu_2\}}$.

This branching multiplicity diagram for [11 4 2] coincides precisely with the weight multiplicity diagram obtained previously for {6 4}. This happy coincidence is not an accident peculiar to the example chosen. Further examples suggest the validity of the formula:

$$\chi^{[\lambda_1\,\lambda_2\,\lambda_3]} = \sum_{\alpha_1=0}^{2\lambda_3} \sum_{\alpha_3=0}^{\lambda_2-\lambda_3} \sum_{\alpha_2=0}^{\alpha_1+\alpha_3} \chi^{\{\lambda_1-\lambda_3+\alpha_1-\alpha_2,\lambda_2-\lambda_3+\alpha_2-\alpha_3\}}. \tag{4.4}$$

This implies that the branching multiplicity diagram of $[\lambda_1\,\lambda_2\,\lambda_3]$ on restriction from SO(7) to G_2 coincides with the SU(3) weight multiplicity diagram for $\{\lambda_2+\lambda_3\;2\lambda_3\}$.

This result may be proved using (4.3) and an induction argument with respect to the parameter λ_1 starting the induction from the initial value $\lambda_1 = \lambda_2$. This particular value corresponds to a special case analysed by Wybourne (1972) for which (4.3) yields, after reflection in the line $\mu_1 = \mu_2 - 1$, a point set bounded by a parallelogram all of whose points have multiplicity 1. This same result is obtained, after reflection, from (4.4).

The necessity of using the modification rule (2.8) along with (4.4) is unavoidable. However for fixed α_1 and α_3 the application of (2.8) to (4.4) gives:

$$\chi^{[\lambda_1\,\lambda_2\,\lambda_3]} = \sum_{\alpha_1=0}^{2\lambda_3} \sum_{\alpha_3=0}^{\lambda_2-\lambda_3} \sum_{\alpha_2=0}^{\lambda_1-\lambda_2} \chi^{\{\lambda_1-\lambda_3+\alpha_1-\alpha_2,\lambda_2-\lambda_3+\alpha_2-\alpha_3\}} \tag{4.5}$$

which is remarkable for its similarity to each of the terms of (4.3). Once again the corresponding point set consists of a set of nested hexagons to which, in general, it is necessary to apply a reflection in order to obtain the final branching rule diagram. To avoid doing this it only remains to note that both (4.4) and (4.5) are, by virtue of (2.8), equivalent to:

$$\chi^{[\lambda_1\,\lambda_2\,\lambda_3]} = \sum_{\alpha_1=0}^{2\lambda_3} \sum_{\alpha_3=0}^{\lambda_2-\lambda_3} \sum_{\alpha_2=0}^{\min(\lambda_1-\lambda_2,\alpha_1+\alpha_3)} \chi^{\{\lambda_1-\lambda_3+\alpha_1-\alpha_2,\lambda_2-\lambda_3+\alpha_2-\alpha_3\}}. \tag{4.6}$$

This is all that can be said on the subject since all the G_2 characters given by the use of this formula are admissible. Application to $[\lambda_1\,\lambda_2\,\lambda_3]$ with $\lambda_1 - \lambda_2 \geqslant \lambda_2 + \lambda_3$ yields the same set of nested hexagons as (4.4) since $\lambda_1-\lambda_2\geqslant\alpha_1+\alpha_3$ whilst application to $[\lambda_1\,\lambda_2\,\lambda_3]$ with $\lambda_1=\lambda_2$ yields the same set of nested parallelograms as (4.5) since $\lambda_1-\lambda_2\leqslant\alpha_1+\alpha_3$. This is exemplified by the branching multiplicity diagram for [4 4 2] as shown in figure 4.

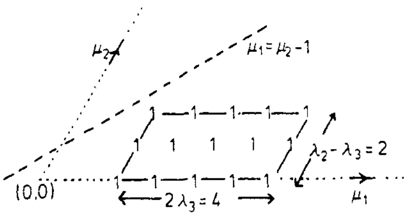

Figure 4. SO(7)$\downarrow G_2$ branching multiplicity diagram for the irreducible representation $[\lambda_1\,\lambda_2\,\lambda_3]=[4\;4\;2]$ illustrating a set of nested parallelograms. The numbers displayed are the branching multiplicities $B^{[4\,4\,2]}_{\{\mu_1\,\mu_2\}}$.

More generally the point set defined by (4.6) is a set of nested heptagons for which the branching multiplicity increases in steps of 1 from the value 1 on the boundary heptagon until the value $1+\min(\lambda_1-\lambda_2,\lambda_2-\lambda_3,2\lambda_3)$ is reached at which stage the 'heptagon' is in fact either a quadrilateral, a triangle, a line or a single point. All points within and on this 'heptagon' have the same multiplicity. This structure is illustrated by the example [7 4 2] of figure 5.

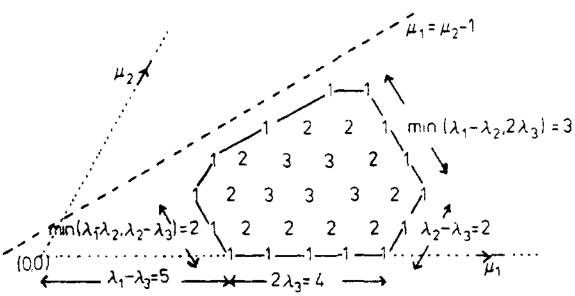

Figure 5. SO(7)$\downarrow G_2$ branching multiplicty diagram for the irreducible representation $[\lambda_1\,\lambda_2\,\lambda_3]=[7\;4\;2]$ illustrating a set of nested heptagons. The numbers displayed are the branching multiplicities $B^{[7\,4\,2]}_{\{\mu_1\,\mu_2\}}$.

The explicit branching multiplicity formula which is the analogue of (3.3) is, in the notation of (4.1) and (3.4):

$$B^{[\lambda_1\,\lambda_2\,\lambda_3]}_{\{\mu_1\,\mu_2\}}=1+\min(\lambda_1-\lambda_2,\lambda_2-\lambda_3,2\lambda_3,\mu_2,\lambda_1-\lambda_3-\mu_2,\lambda_2+\lambda_3-\mu_2,\mu_1-\mu_2,\mu_1+\mu_2-\lambda_1+\lambda_3,\mu_1-\lambda_1+\lambda_2,\mu_1-\lambda_2+\lambda_3,\lambda_1+\lambda_2-\mu_1-\mu_2,\lambda_1+\lambda_3-\mu_1). \tag{4.7}$$

The last nine arguments of min(. . .) are the distances of the point $(\mu_1\,\mu_2)$ from each of the possible edges of the boundary of the branching multiplicity diagram.

5. Conclusion

The final result (4.6) is the best possible in the sense that it not only involves no further use of the modification rule (2.8) but also yields the explicit formula (4.7). It is remarkable both for the ease with which it reproduces (and corrects where appropriate) all the branching rules given by Wybourne (1972) for a variety of classes of special cases and for the prediction that for $\lambda_1-\lambda_2 \geqslant \lambda_2+\lambda_3$ the branching multiplicity diagram for $[\lambda_1\,\lambda_2\,\lambda_3]$ is precisely the weight multiplicity diagram for $\{\lambda_2+\lambda_3\;2\lambda_3\}$.

Finally it should be pointed out that the validity of (4.6) proves the completeness, conjectured by Wybourne (1972), of a particular set of elementary multiplets appropriate to the restriction from SO(7) to G_2. Wybourne's conjecture corresponds to the branching formula:

$$\chi^{[\lambda_1\,\lambda_2\,\lambda_3]} = \sum_{a,b,\ldots,f} \chi^{\{b+c+d+e+f+g,e+f+g\}} \tag{5.1}$$

with $\lambda_1-\lambda_2=c+f+g$, $\lambda_2-\lambda_3=d+e+g$, $2\lambda_3=a+b+f$, $df=0$. The terms in this expression are in one-to-one correspondence with those of (4.6) as established by the relations $a=2\lambda_3-\alpha_1$, $c=\lambda_1-\lambda_2-\alpha_2$, $e=\lambda_2-\lambda_3-\alpha_3$, together with either $b=\alpha_1$, $d=\alpha_3-\alpha_2$, $f=0$, $g=\alpha_2$ or $b=\alpha_1$, $d=0$, $f=0$, $g=\alpha_2$ or $b=\alpha_1-\alpha_2+\alpha_3$, $d=0$, $f=\alpha_2-\alpha_3$, $g=\alpha_3$ according as either $0\leqslant\mu_2<\lambda_2-\lambda_3$ or $\mu_2=\lambda_2-\lambda_3$ or $\lambda_2-\lambda_3<\mu_2$.

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